

A STANDARD FORMULATION OF THE STEADY CONVECTIVE MASS TRANSFER PROBLEM

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Abstract—It is shown that, provided that certain relations exist between the transport properties, every mass transfer problem, including those with simultaneous heat transfer and chemical reaction, can be expressed by relations of the "Ohm's Law" type: $\dot{m}'' = g \cdot B$, where \dot{m}'' is the required mass transfer rate through the surface, g is a surface conductance dependent on aerodynamic factors, and B is a dimensionless driving-force dependent on the thermodynamic properties of the main stream, of the fluid in contact with the phase boundary, and of the transferred substance.

The standard formulation is derived from the differential equations of conservation and flux and by reference to a wall-flux boundary condition. Thereafter examples are given of methods of calculating g , followed by examples of expressions for B valid for particular cases.

Résumé—A condition que certaines relations existent entre les propriétés de transport, on montre que tout problème de transport de masse, même s'il comprend à la fois transfert de chaleur et réaction chimique, peut être exprimé par des relations du type de la loi d'Ohm: $\dot{m}'' = g \cdot B$ où \dot{m}'' est la densité de flux de masse transportée à travers la surface, g est la conductance de surface qui dépend des facteurs aérodynamiques et B est la force agissante, sous forme adimensionnelle, qui dépend des propriétés thermodynamiques de l'écoulement principal, du fluide en contact avec la surface de séparation, et de la substance transportée.

La formulation générale est obtenue à partir des équations aux dérivées partielles de conservation et de transport, rapportées aux conditions aux limites du transports à la paroi.

Des exemples sont donnés ensuite pour les méthodes permettant le calcul de g , et sont suivis par différentes expressions de B valables dans des cas particuliers.

Zusammenfassung—Es wird gezeigt, dass jedes Problem der Stoffübertragung einschliesslich derer mit gleichzeitiger Wärmeübertragung und chemischer Reaktion in Form eines Ohm'schen Gesetzes $\dot{m}'' = g \cdot B$ ausgedrückt werden kann, sofern gewisse Beziehungen zwischen den Transportgrössen bestehen. Es bedeutet \dot{m}'' die gesuchte Mengestromdichte an der Oberfläche, g eine Oberflächenleitfähigkeit, abhängig von aerodynamischen Faktoren und B eine dimensionslose treibende Kraft, die von den thermodynamischen Eigenschaften des Hauptstroms, der Flüssigkeit im Kontakt mit der Phasengrenzfläche und des übertragenen Stoffes abhängt. Aus den Differentialgleichungen der Erhaltung und der Strömung unter Bezugnahme auf eine Grenzbedingung der Wandströmung wurde eine einheitliche Formulierung abgeleitet. Beispiele für die Methoden zur Berechnung von g werden mitgeteilt und schliesslich werden die Ausdrücke von B für besondere Fälle angegeben.

Abstract—Показано, что при условии существования определенных соотношений между характеристиками переноса каждая задача переноса, включая также задачи с одновременно происходящими химической реакцией и переносом тепла, может быть выражена при помощи соотношений, подобных закону Ома: $\dot{m}'' = g \cdot B$, где \dot{m}'' —требуемая скорость переноса массы через поверхность, g —поверхностная проводимость, зависящая от аэродинамических факторов, B —безразмерная движущая сила, зависящая от термодинамических свойств основного потока, жидкости, соприкасающейся с поверхностью раздела фаз, а также переносимой субстанции.

Общая постановка задачи получена из анализа дифференциальных уравнений сохранения и потока с учётом условий на границе сменного потока. Приводятся методы вычисления g , сопровождаемые примерами выражений для B , справедливых для частных случаев.

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NOTATION

b ,	dimensionless transferred property, equation (28);	\dot{m}''_j ,	volumetric rate of generation of substance j by homogeneous chemical reaction (lb/ft ³ hr);
B ,	dimensionless driving force for mass transfer ($\equiv b_G - b_S$), equation (33);	n_a ,	mass of chemical element a in unit mass of mixture (dimensionless);
c ,	specific heat of fluid mixture at constant pressure (B.t.u./lb °F);†	$n_{a,j}$,	mass of chemical element a in unit mass of substance j (dimensionless);
D ,	diffusion coefficient (ft ² /hr);	P ,	conserved property, equation (26);
f ,	mass fraction in fluid mixture of one component in a two-stream flow, regardless of state of chemical aggregation (dimensionless);	P_1 ,	conserved property, equation (16);
F ,	function of B and μ/γ appearing in equation (36);	P_2 ,	conserved property, equation (20);
g ,	surface conductance for mass transfer, equations (34) and (34a), (lb/ft ² hr);	P_3 ,	conserved property, equation (21);
g^* ,	value of g when $B = 0$, equation (45), (lb/ft ² hr);	\dot{q}''_L ,	heat flux away from considered phase across the L control surface, Fig. 3 (B.t.u./ft ² hr);
G ,	vectorial total mass flux at any point in fluid (lb/ft ² hr);	\dot{q}''_S ,	heat flux away from considered phase across the S control surface, Figs. 1 and 3, (B.t.u./ft ² hr);
G ,	"mass velocity" at reference point in stream (lb/ft ² hr);	Q ,	heat flux across S per unit mass transfer, equation (54), (B.t.u./lb);
h ,	specific enthalpy of mixture ($\equiv \sum m_j h_j$), (B.t.u./lb);	r ,	mass of "oxidant" combining with unit mass of "fuel" in simple chemical reaction (dimensionless);
h_j ,	partial enthalpy of substance j (B.t.u./lb);	R ,	distance of location on surface from axis of symmetry, equation (36) (ft);
h_{LS} ,	"latent heat" of phase change of transferred substance (B.t.u./lb);	t ,	temperature (°F);
H ,	heat of reaction of "fuel" at constant-pressure (B.t.u./lb);	x ,	distance along surface in stream-wise direction (ft);
k ,	thermal conductivity of mixture (B.t.u./ft hr °F);	y ,	distance normal to surface (ft);
K ,	function of B and μ/γ appearing in equation (36), equal to ordinate of Fig. 2;	α ,	surface heat transfer coefficient (B.t.u./ft ² hr °F);
l ,	typical dimension of body (ft);	γ ,	"exchange coefficient", $= D\rho$ (lb/ft hr);
m_j ,	mass of j in unit mass of mixture (dimensionless);	Γ ,	"thermal exchange coefficient" $\equiv k/c$ (lb/ft hr);
\dot{m}'' ,	total mass flux across phase boundary into considered phase (lb/ft ² hr);	μ ,	dynamic viscosity (lb/ft hr).
$\dot{m}''_{j,S}$,	mass flux of substance j across control surface S into considered phase (lb/ft ² hr);	Subscripts	
\dot{m}''_a ,	mass flux of chemical element a into considered phase (lb/ft ² hr);	a ,	chemical element;
		i ,	inert chemical compound;
		j ,	chemical compound;
		fu ,	reactant in a simple reaction, e.g. fuel;
		ox ,	reactant in a simple reaction, e.g. oxidant;
		$prod$,	product in a simple reaction, e.g. product;
		$steel$,	steel;
		$cool$,	coolant gas in transpiration-cooling;
		Fe ,	iron;
		C ,	carbon;

† The abbreviation for the pound mass is lb, and for the pound force Lb throughout the paper.

G ,	state of fluid mixture in main stream;
S ,	state of fluid mixture at S control surface;
T ,	state of transferred substance, see Section 2.2;
L ,	state of fluid mixture at L control surface;
O ,	state of transpiration coolant in supply reservoir;
0 ,	reference state;
e ,	main stream at outer "edge" of boundary layer.

the fluid in contact with the surface, and of the transferred substance;
 g is a surface conductance (units = lb/ft²hr) expressing the influence of fluid-mechanical factors: stream velocity, surface shape, etc.

1. INTRODUCTION

1.1. Purpose of the paper

THERE are many methods available for the calculation of mass transfer rates in particular circumstances. The notation, generality, and the physical and mathematical exactness of the methods currently differ according to the industry in which the problem has arisen. Thus, to name just two examples, chemical engineers calculating absorption rates use molal concentrations, the "stagnant-film" idealization, and an empirical equation for the surface conductance; while aeronautical engineers concerned with transpiration cooling use mass concentrations and exact or approximate theoretical solutions of the laminar boundary-layer equations.

It is the aim of the present paper to outline a formulation of the mass transfer problem which is sufficiently flexible and general to accommodate all mass transfer processes and sufficiently simple in use to be accessible to design engineers.

1.2. Outline of the proposed method

The Ohm's law of mass transfer. The aim of convective mass transfer theory is to calculate the rate of transfer of material across a stream boundary. We give this quantity the symbol \dot{m}'' , with units: lb/ft²hr.

The laws governing mass transfer processes permit the mass transfer rate to be related to the stream and boundary properties by an equation of the Ohm's law form, namely:

$$\dot{m}'' = g \cdot B \quad (34)$$

where B is a dimensionless driving force, dependent for its value on the composition and temperature of the main stream, of

Methods of calculating the conductance g are discussed in Section 2.5 below. The whole of Section 3 is devoted to the evaluation of the driving-force B in particular circumstances.

Range of applicability of the method. The method is valid for both gases and liquids in laminar or turbulent steady flow. Chemical reaction may occur. In the latter case, however, the permissible transport-property variations are restricted. These restrictions are discussed in Section 2.3.

1.3. Relation to earlier work

Origins. The present method arose out of studies of the work of Hatta [1] on liquid-phase reactions, as presented in the book by Sherwood [2], and of the work of Nusselt [3] and Burke and Schumann [4] on the combustion of carbon. The emphasis on concordance with boundary-layer theory was stimulated by the work of Eckert and Lieblein [5] but, by a fruitful accident, mass concentrations were used rather than molal ones.

The first presentations of the general method, using the differential equations of atomic and energy conservation [6] and with derivation of corresponding approximate solutions of the laminar boundary-layer equations [7] were made in connexion with studies of the combustion of liquid fuels. The first exact solutions of the boundary-layer equations in the present terms were published by Emmons [8]. The formulation was later somewhat elaborated [9, 10] particularly in the direction of the use of constructions on enthalpy-composition diagrams for the solution of simultaneous-transfer problems [11, 12, 13]; the latter are not discussed here.

The present paper represents a development of the above work, being improved as regards rigour of derivation, note having been taken of the criticisms of Merk [14]. A more careful

definition of the concept of the "transferred substance" has permitted extension of the method to processes, common in chemical-engineering equipment, in which substances are transferred through the phase boundary in both directions simultaneously.

Parallel work. The ideas and techniques which will be presented have also been independently invented and exploited by many other authors. It is only possible to mention the main contributors here.

An important stream of related work is that started by Busemann [15] and continued by Bosnjakovic [16-20]. The latter author has made particularly extensive use of graphical methods for transfer calculations, and has made a thorough study of the simultaneous processes of mass transfer, heat transfer and chemical reaction which occur in the gas producer.

The simplifications which can be made to the differential equations when equality exists between the diffusivities of matter and heat have long been exploited in the theory of gaseous flames. Particular mention must be made of the early contributions in this regard by the Russian workers, Zeldovich and Frank-Kamenetsky [21] and Schwab [22]. These authors studied the system designated "simple chemical reaction" below (simplifying assumptions (iii), (iv), (v), (vii), (viii) and (ix)), but did not consider mass transfer across a phase boundary.

Since World War II, numerous theoretical studies have been made of the combustion of liquid-fuel droplets in oxidizing atmospheres; workers who use methods similar to the author's include Varshavskii [23], Goldsmith and Penner [24], and Agafanova *et al.* [25]. A generalization has been made by Coffin and Brokaw [26].

Still more recently, interest in simultaneous mass transfer and chemical reaction has been awakened among aerodynamicists concerned with the cooling of the surfaces of high-speed missiles. The most extensive theoretical study of this subject is that of Lees [27]. Related publications are those of Sutton [28], Hartnett and Eckert [29], Bromberg and Lipkis [30], and others too numerous to mention.

Remarks on notation. The notation used by the above authors naturally exhibits wide deviations from that used here, and cannot be

reviewed in detail. However, the following remarks may be of assistance.

In the chemical-engineering literature (see, for example, Sherwood and Pigford [31]), the symbol k_G for gas-side surface conductance is equivalent to g , used here, divided by (absolute pressure times local mixture molecular weight). The symbols k_L and k_c are equivalent to g divided by local mixture density. Many of the symbols for surface conductance used by other authors have the latter significance.

Our surface conductance g is equal to the surface heat transfer coefficient a divided by the constant-pressure specific heat of the mixture c , if enthalpy transfer is in question. Therefore $g l/(k/c)$ has the significance of a Nusselt number, where l is a reference dimension of the body and k is the mixture thermal conductivity. The writer finds it preferable to introduce the symbol Γ for k/c , and the symbol γ for $D\rho$ where D is the diffusion coefficient and ρ the local density.

The driving force B , was referred to as the "transfer number" in the writer's previous work. The symbol B' used by Lees [27] has the same significance.

In the present work the mass fraction, m_j , will be used as the measure of concentration, regardless of the phase of the mixture. Thus, in the present notation, m_j signifies the mass of chemical compound j to be found in unit mass of local mixture.

1.4. Guide to the remainder of the paper

Sections 2.1, 2.2, 2.3 and 2.4 contain, in condensed form, the analysis leading from the laws of physics to the standard differential equation and boundary condition of mass transfer; the various sets of essential simplifying assumptions appear in Section 2.3. *All these sections can be skipped by readers interested only in the use of the method.*

Section 2.5 explains how the surface conductance g can be calculated in a variety of practical cases; the formulae are stated but not derived. Section 2.6 then summarizes the calculation procedure which is recommended for predicting mass transfer rates.

Section 3 shows, by way of a series of examples, the manner in which the driving force

B can be calculated. The examples are chosen so as to cover processes arising in chemical, aeronautical and mechanical engineering.

2. SIMPLIFICATION, STANDARD FORMULATION, AND SOLUTION OF THE DIFFERENTIAL EQUATIONS

2.1. Differential equations of conservation and flux

The experimental laws governing the distribution of (time—mean) concentration and enthalpy in a non-uniform steady fluid stream can be represented by differential equations. The laws comprise the conservation of matter, the First Law of Thermodynamics, the Fick Diffusion Law, and the Fourier Law of Heat Conduction.

“Coupling” of the fluxes is disregarded, i.e. it is supposed that no diffusion of mixture-component j occurs in the absence of a gradient in the mass fraction of j , m_j , and that no heat transfer occurs in the absence of a temperature gradient. The barycentric co-ordinate system is used; this means that the diffusion flux of mixture-component j is defined as equal to the total mass flux vector of j at the point in question minus the convective mass flux vector of j ; the latter quantity is defined as $m_j \mathbf{G}$, where m_j is the local mass fraction of j , while \mathbf{G} is the net mass flux vector, i.e. the sum of the total mass flux vectors of all the components of the mixture.

The equations are written in vector form for the sake of compactness;† they are:

† Some readers may like to be reminded that, in Cartesian form, equation (1), for example, would read:

$$\rho u \frac{\partial m_i}{\partial x} + \rho v \frac{\partial m_i}{\partial y} + \rho w \frac{\partial m_i}{\partial z} - \frac{\partial}{\partial x} \left(\gamma_i \frac{\partial m_i}{\partial x} \right) - \frac{\partial}{\partial y} \left(\gamma_i \frac{\partial m_i}{\partial y} \right) - \frac{\partial}{\partial z} \left(\gamma_i \frac{\partial m_i}{\partial z} \right) = 0$$

where x , y and z are the Cartesian co-ordinate directions; u , v and w are the local fluid velocities in these directions, defined so that, if ρ is the local fluid density, then

ρu , ρv and ρw are the components of the total mass flux vector \mathbf{G} .

The other equations can be similarly translated; although the presence of the summations, \sum_j , renders them rather unwieldy.

for a chemically inert substance, i

$$\mathbf{G} \cdot \nabla(m_i) - \nabla \{ \gamma_i (\nabla m_i) \} = 0 \quad (1)$$

for a substance j , which may enter into a chemical reaction

$$\mathbf{G} \cdot (\nabla m_j) - \nabla \{ \gamma_j (\nabla m_j) \} = \dot{m}'''_j \quad (2)$$

where \dot{m}'''_j is the rate of generation of substance j by chemical reaction (units, lb/ft³hr)

For a chemical element α :

$$\mathbf{G} \cdot (\nabla n_\alpha) - \nabla \left\{ \sum_j \gamma_j n_{\alpha,j} (\nabla m_j) \right\} = 0 \quad (3)$$

where $n_{\alpha,j}$ is the mass of chemical element α contained in unit mass of chemical compound j ;

n_α is the mass of chemical element α contained in unit mass of local mixture.

For the enthalpies, when shear work,‡ kinetic energy, gravitational potential energy and electrical and magnetic effects can be neglected:

$$\mathbf{G} \cdot (\nabla h) - \nabla \left\{ \sum_j \gamma_j h_j (\nabla m_j) \right\} - \nabla \{ \Gamma c (\nabla t) \} = 0 \quad (4)$$

where h_j is the partial enthalpy of component j in the local mixture (B.t.u./lb);

h is the specific enthalpy of the mixture ($\equiv \sum_j h_j m_j$).

In equation (4), the first term represents the convective enthalpy flux, the second term represents the enthalpy flux associated with diffusion, while the third represents the heat flux (since $\Gamma c \equiv k$, the thermal conductivity. The reason for preferring the use of Γ should become apparent later when the simplified forms of the equation are dealt with).

2.2. The wall-flux boundary condition

Consider an element, illustrated in Fig. 1, of a surface separating the fluid under consideration from a neighbouring phase; this surface can be termed variously “phase boundary”, “interface”

‡ At high velocities, the enthalpy h in the first term must be replaced by the stagnation enthalpy of the mixture, and a further term must be added expressing the shear work per unit volume. Details are given in the Appendix.

or "wall". The surface is pervious to material; we shall relate the magnitude of the mass flux through it, \dot{m}'' , to the gradients in stream properties measured at the control surface S which is in the fluid immediately adjacent to the interface. To do this we must consider two states: the S -state and the T -state.

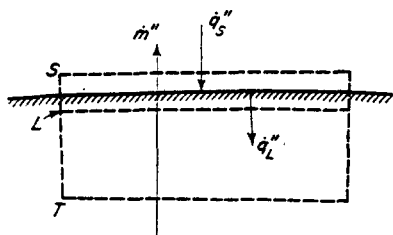


FIG. 1. Control volume used in deriving the boundary conditions, equations (9), (10) and (11).

The S -state is the state (time-mean composition, temperature, etc.) of the fluid mixture at the control surface S . The quantities $m_{j,S}$, t_S , $h_{j,S}$, etc., can in principle be measured by the usual sampling and analysis instruments.

The T -state, i.e. the state of the *transferred substance*, may be regarded as a fictitious mixture state: it does not exist at any particular point, but is defined as follows. Its *composition* is sufficiently defined in terms of the mass fractions $m_{i,T}$, $n_{a,T}$, etc., via the equations involving mass fluxes:

$$m_{i,T} \equiv \dot{m}''_i / \dot{m}'' \quad (5)$$

where \dot{m}''_i is the total (i.e. convective plus diffusive) mass flux of the inert substance i into the fluid phase; and

$$n_{a,T} \equiv \dot{m}''_a / \dot{m}'' \quad (6)$$

where \dot{m}''_a is the total mass flux of the chemical element a into the fluid phase, irrespective of the chemical compounds in which a happens to find itself.

The state of chemical aggregation of the transferred substance is not completely specified by these definitions; it is not required. The quantity \dot{m}''_a is conveniently evaluated in terms of the total mass fluxes of the various substances, $\dot{m}''_{j,S}$, through the control surface S , by way of equation:

$$\dot{m}''_a = \sum_j n_{a,j} \dot{m}''_{j,S} \quad (7)$$

In contrast to the practice for \dot{m}''_i and \dot{m}''_a , a suffix (here S) has to be appended to \dot{m}''_j because, in general, j may take part in a chemical reaction; thus an evaluation of \dot{m}''_j at the control surface L of Fig. 3 would, in general, yield a different value from an evaluation at S . But of course

$$\sum_j n_{a,j} \dot{m}''_{j,S}$$

must be equal to

$$\sum_j n_{a,j} \dot{m}''_{j,L}$$

because chemical elements are neither created nor destroyed. Evaluation at the S control surface is preferable for present purposes because this surface lies in the fluid under consideration.

The *enthalpy of the transferred substance*, h_T , is defined for similar reasons by the equation:

$$h_T \dot{m}'' \equiv \sum_j h_{j,S} \dot{m}''_{j,S} - \dot{q}''_S \quad (8)$$

where \dot{q}''_S is the heat flux caused by the temperature gradient across the control surface S , directed outward from the fluid under consideration.

The T -state may thus be interpreted as that which must prevail at the lower surface of the fictitious control volume of Fig. 1 if the conservation laws are to hold and if there is no diffusion or heat transfer across this surface.

Application of the conservation and flux laws underlying equations (1), (3) and (4) to the control volume of Fig. 1 now yields the following boundary conditions:

For the chemically inert substance i :

$$\dot{m}'' = \frac{\{\gamma_i (\partial m_i / \partial y)\}_S}{m_{i,S} - m_{i,T}} \quad (9)$$

where y is the distance normal to the wall.

For the chemical element a :

$$\dot{m}'' = \frac{\sum_j \{\gamma_j n_{a,j} (\partial m_j / \partial y)\}_S}{n_{a,S} - n_{a,T}} \quad (10)$$

For the enthalpy:

$$\dot{m}'' = \frac{\sum_j \{\gamma_j h_j (\partial m_j / \partial y)\}_S + \{Tc (\partial t / \partial y)\}_S}{h_S - h_T} \quad (11)$$

No such boundary condition is available in general for the chemical substance j since the

state of chemical aggregation of the transferred substance is not specified (j may react at the phase boundary).

2.3. Simplifications necessary if reaction-kinetic considerations are to be avoided

Because of the term appearing on the right-hand side of equation (2), evaluation of which involves knowledge of the reaction kinetics of the fluid phase, solution of the equations (3) and (4), with boundary conditions (10) and (11), is not possible unless simplifying assumptions are made. The validity of these assumptions will not be examined here, but it should be remarked that those relating to the transport coefficients are close to reality for all fully turbulent fluids and for gases containing components of roughly uniform molecular weight. The simplifying assumptions represent alternatives.

Simplifying assumption (i). When, at any particular point in the stream, all the γ_j 's of substances containing element α are equal to each other, equations (3) and (10) reduce to:

$$G. (\nabla n_\alpha) - \nabla \{\gamma_\alpha (\nabla n_\alpha)\} = 0 \quad (12)$$

and

$$\dot{m}'' = \frac{\{\gamma_\alpha (\partial n_\alpha / \partial y)\}_S}{n_{\alpha,S} - n_{\alpha,T}} \quad (13)$$

where γ_α is the common value of the set of γ_j 's.

Simplifying assumption (ii). When, at any particular point in the stream, all the γ_j 's are equal to each other and to the quantity Γ , equations (4) and (11) reduce to:

$$G. (\nabla h) - \nabla \{\Gamma (\nabla h)\} = 0 \quad (14)$$

and

$$\dot{m}'' = \frac{\{\Gamma (\partial h / \partial y)\}_S}{h_S - h_T} \quad (15)$$

Simplifying assumption (iii). Suppose that two substances, designated "fuel" and "oxidant" for concreteness, take part in a simple chemical reaction to produce a third substance (product) without intermediate transformations; suppose further that the quantities γ_{fu} and γ_{ox} are equal to each other at every point in the stream then, if we define a quantity P_1 , through:

$$P_1 = m_{fu} - m_{ox}/r \quad (16)$$

where r is the mass ratio of oxidant to fuel in the stoichiometric equation, then the recognition that, for this reaction, we can put:

$$\dot{m}''_{fu} = \dot{m}''_{ox}/r \quad (17)$$

permits us to deduce from equation (2) that:

$$G. (\nabla P_1) - \nabla \{\gamma_1 (\nabla P_1)\} = 0 \quad (18)$$

The quantity $P_{1,T}$ can be evaluated, since it does not entail knowledge of the state of chemical aggregation of the transferred substance. Consideration of the control volume of Fig. 1 now yields the boundary condition:

$$\dot{m}'' = \frac{\{\gamma_1 (\partial P_1 / \partial y)\}_S}{P_{1,S} - P_{1,T}} \quad (19)$$

γ_1 is of course the common γ of the substances designated "fuel" and "oxidant". Obviously, despite the use of the latter terms, the analysis is in no way restricted to combustion reactions.

As alternatives to the assumptions leading to (18) and (19), two further cases may be considered, namely:

(iv) If the γ 's of fuel and product are equal to each other at every point in the stream, having the common value γ_2 ; and if a quantity P_2 is defined by:

$$P_2 \equiv m_{fu} + m_{prod}/(1 + r) \quad (20)$$

then equations (18) and (19) hold with the subscript (2) in place of subscript (1).

(v) If the γ 's of oxidant and product are equal (to γ_3) at each point, and P_3 is defined by:

$$P_3 = m_{ox}/r + m_{prod}/(1 + r) \quad (21)$$

then the equations (18) and (19) hold with the subscript (3) in place of subscript (1).

Simplifying assumption (vi). When chemical reaction is absent, the fluid mixture is ideal, and the specific heat of the fluid is uniform, equations (4) and (11) again reduce to (14) and (15). The enthalpy h can in this case be expressed in terms of temperature, at any rate within the phase under consideration, via:

$$h \equiv c(t - t_0) \quad (22)$$

where t_0 is an arbitrary reference temperature. No restriction on the γ 's need be imposed.

Simplifying assumption (vii). When a single

simple chemical reaction takes place, of the type described under (iii), if the fluid mixture is otherwise ideal with uniform specific heat, and if further the γ of fuel is equal to Γ at every point in the mixture, equations (4) and (11) again reduce to equations (14) and (15). This time the fluid-phase enthalpy can be expressed as:

$$h \equiv m_{fu} H + c(t - t_0) \quad (23)$$

where H is the heat of reaction per unit mass of fuel.

Alternative simplifications are clearly possible, as follows:

(viii) If, instead of $\gamma_{fu} = \Gamma$, we have $\gamma_{ox} = \Gamma$ at every point, equations (14) and (15) now hold with:

$$h \equiv m_{ox} H/r + c(t - t_0) \quad (24)$$

(ix) If, alternatively, $\gamma_{prod} = \Gamma$, equations (14) and (15) hold with the fluid-phase enthalpy defined through:

$$h \equiv -m_{prod} H/(1 + r) + c(t - t_0) \quad (25)$$

Further simplifying assumptions. The above nine cases do not exhaust the possibilities. Other simplifications involving a different mixture of restrictions on the transport properties and restrictions on the thermodynamic properties can be easily envisaged; they may be convenient in the analysis of particular cases.

Generalization. Comparison of the pairs of equations (1) and (9), (12) and (13), (14) and (15), (18) and (19), shows that they have a common form which we may write as:

$$G \cdot (\nabla P) - \nabla \{ \gamma (\nabla P) \} = 0 \quad (26)$$

and

$$\dot{m}'' = \frac{\{ \gamma (\partial P / \partial y) \}_S}{P_S - P_T} \quad (27)$$

where $P \equiv m_i, n_a, h, P_1, P_2, P_3$, or any other conserved property,[†] and γ may be replaced by Γ if enthalpy is in question.

[†] Fluid properties obeying equation (26) might be called "conserved properties of the second class" to distinguish them from "conserved properties of the first class" which are conserved in adiabatic steady-flow mixing processes.

2.4. Re-arrangement for the case of uniform S- and T-states

All the solutions of the equation (26) which are known to the author (except those for $\dot{m}'' = 0$) relate to cases in which the state of the fluid in contact with the surface *and* the state of the transferred substance are uniform over the boundaries of the field of integration through which mass transfer occurs. This means that P_S and P_T are constants in the integration procedure.

In such cases it is convenient to reduce all mass transfer problems to a common form by introducing the new variable b , defined as:

$$b \equiv \frac{P - P_S}{P_S - P_T} \quad (28)$$

Equations (26) and (27) then reduce to:

$$G \cdot (\nabla b) - \nabla \{ \gamma (\nabla b) \} = 0 \quad (29)$$

$$\dot{m}'' = \left(\gamma \frac{\partial b}{\partial y} \right)_S \quad (30)$$

Nature of the mathematical problem. We now see that, whatever the nature of the conserved property which is under consideration, the task of predicting mass transfer rates reduces to the solution of equation (29) with, as boundary conditions at the wall, equation (30) and:

$$b_S = 0 \quad (31)$$

Since G and γ appear in (29), solution of this equation requires prior (or sometimes simultaneous) solution of the momentum and continuity equations for the fluid stream, and also a proper accounting for the dependence of γ on the fluid state, i.e. on b .

Behaviour of the solution. Without treating any particular configuration of surface and stream, it is possible to conclude, from dimensional analysis and other general considerations, that the solution to any forced-convection mass transfer problem with uniform S- and T-states can be cast in the form:

$$\frac{\dot{m}''}{G} = f \left\{ B, \frac{Gl}{\mu_0}, \frac{\mu_0}{\gamma_0} \right\} \quad (32)$$

where G is the mass velocity in the main stream, say at entry;

l is an apparatus dimension;
 μ_0 is the fluid viscosity measured at some reference state;
 γ_0 is the corresponding value of the exchange coefficient of the relevant P ;
 $B \equiv b_G$, the value of b in the main stream.

The latter definition leads to the important relation:

$$B \equiv \frac{P_G - P_S}{P_S - P_T} \quad (33)$$

Further thought about the solution reveals that it may be convenient to introduce a quantity g (units, lb/ft²/hr) via the definition:

$$m'' \equiv gB \quad (34)$$

and to express the solution to equation (29) in the form:

$$\frac{g}{G} = f \left\{ B, \frac{Gl}{\mu_0}, \frac{\mu_0}{\gamma_0} \right\} \quad (35)$$

The quantity on the left-hand side of (35) is akin to the *Stanton* number; Gl/μ_0 is a *Reynolds*

number; and μ_0/γ_0 is a Prandtl or Schmidt number according to the nature of the conserved property appearing in B . Alternatively, the group, gl/γ_0 a sort of Nusselt number, may appear on the left-hand side.

By comparing equations (30) and (34), we see that g is obtained from the solution of equation (29) via:

$$g = \left\{ \gamma \frac{\partial b}{\partial y} \right\}_s / B \quad (34a)$$

2.5. Some solutions or calculation procedures for g

Solution for the flat plate in laminar flow. As an example, Fig. 2 presents a plot, based on the exact boundary-layer solutions of Mickley *et al.* [32], of $(gx/\gamma)/\sqrt{(G_e x/\mu)}$ vs. B for the flat plate in a laminar stream. The transport properties μ and γ are supposed uniform; x is the distance from the leading edge at which g prevails, and G_e is the uniform mass velocity of the stream. Each curve is valid for a particular μ/γ (Prandtl or Schmidt number).

Exact solutions are not available for very high or very low values of μ/γ . However, for high

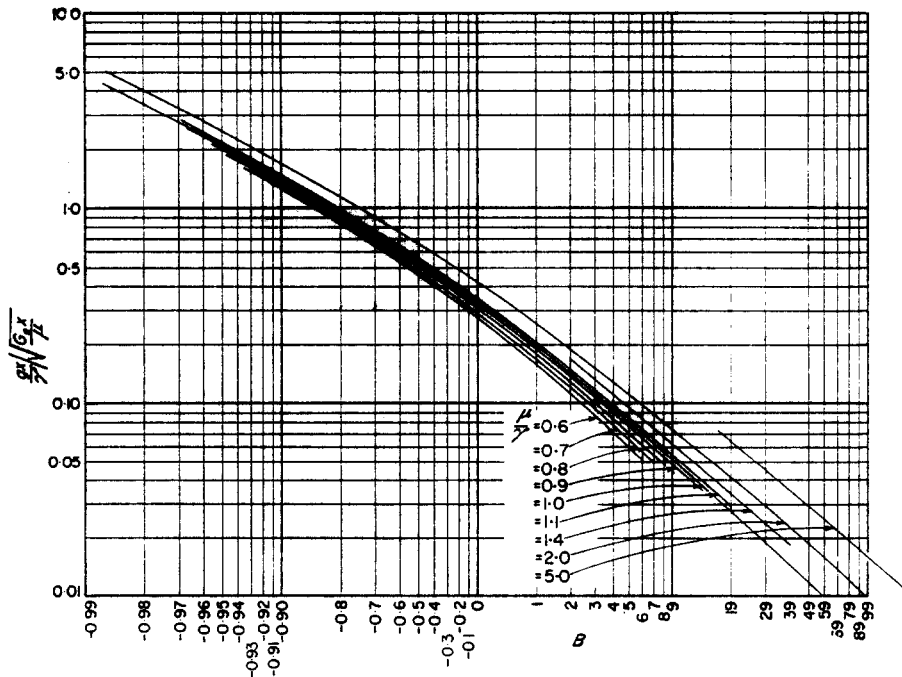


FIG. 2. Exact solutions of the boundary-layer equations for the flat plate in laminar flow [32].

values with B lying between -0.5 and $+1.0$, the asymptotic solution of Merk [33] can be used for the ordinate of Fig. 2: it is

$$0.3387 (\mu/\gamma)^{1/3} / (1 + 0.566 B).$$

Other solutions for laminar boundary-layer flows. Surprisingly, only a handful of exact solutions of the laminar boundary-layer equations are available for other cases than the flat plate. Spalding and Smith [41] have recommended that mass transfer through two-dimensional or axi-symmetrical laminar boundary layers in general should be calculated from an approximate quadrature formula which, in the present terms, runs:

$$g = K(\gamma/\mu^{1/2}) \left(R^2 G_e^F / \int_0^x R^2 G_e^{F-1} dx \right)^{1/2} \quad (36)$$

where x is the stream-wise distance from the location at which mass transfer begins;

R is the distance of the section in question from the axis of symmetry (N.B. the R 's disappear for a two-dimensional flow);

G_e is the mass velocity at the outer "edge" of the boundary layer at location x ;

g is the surface conductance, defined by (34), valid for location x ;

K is a function of B and μ/γ (in fact it is equal to the ordinate of Fig. 2);

F is a function of B and μ/γ .

Interpolating in exact solutions tabulated by Livingood and Donoughe [35], Spalding and Smith have deduced some values of K and F valid for $\mu/\gamma = 0.7$; these values are contained in Table 1. At other values of μ/γ , K may be obtained from Fig. 2; F tends to 3.8 for all B at high μ/γ .

Table 1

B	0	0.5	1.0	2.0	3.0	4.0	5.0
K	0.292	0.213	0.169	0.120	0.0930	0.0770	0.0670
F	2.68	3.07	3.43	3.73	4.04	4.36	4.43

There is a great need for extended tables of K and F , covering both positive and negative B values and a wide range of μ/γ . Thereafter the influence of variable μ and γ needs to be thoroughly investigated; only a few isolated studies of this effect have been made so far. Efforts to fill these gaps are being made in the writer's laboratory.

Calculation method for mass transfer through turbulent boundary layers. Still fewer studies have been made of mass transfer through turbulent boundary layers. A quadrature formula which is simple, and probably fairly reliable, can be derived by extending the method of Ambrok [36] to mass transfer; one obtains for $\mu/\gamma = 0.7$

$$\frac{g}{G_e} = 0.0365 \left[\frac{\ln(1 + B)}{B} \right] \left[\frac{\mu R^{5/4}}{\int_0^x R^{5/4} G_e dx} \right]^{1/5} \quad (37)$$

Once again there is a great need for experimental and theoretical research in this field; important contributions are being made by Mickley and co-workers [32, 37], and by Rubesin and co-workers [38, 39].

Other methods. Of course the value of g need not be obtained by theoretical analysis at all: it may be obtained from experiment. The book of Sherwood and Pigford [31] contains many such experimentally derived values, most of them obtained with B values close to zero, and with almost uniform transport properties (but see "remarks on notation" above). The value of g may also be obtained by way of the Reynolds analogy; it is equal to the shear stress divided by the stream velocity. This value is of course usually only approximate, as is well known.

2.6. General procedure for calculating mass transfer rates

The practical importance of the way in which the mass transfer problem has been formulated above will now be summarized:

(a) The influences of the shape of mass transfer surface, the fluid velocity, the transport properties, etc., may be determined by aerodynamic analysis; this results in graphs or formulae of the type just presented. These graphs or formulae are independent of the

nature of the transfer process, i.e. they hold whether the process comprises vaporization, combustion, transpiration-cooling, distillation or any other process.

(b) The influences of the particular process are expressed through the value of the driving force B , defined by equation (33). B depends only on conditions in the G - (main stream), S - and T -states, and not on the aerodynamics. Examples of the evaluation of B are given below.

(c) With B obtained from the thermodynamic analysis of (b) and g obtained from the aerodynamic analysis of (a), the required mass transfer rate is obtained from equation (34).

(d) Separation of the aerodynamic and thermodynamic aspects is unfortunately not quite complete, a connexion is effected when the variation of the transport properties μ and γ with temperature, etc., is appreciable. This difficulty can be partially overcome by judicious use of "average" values of the transport properties.

3. SOME DRIVING-FORCE EXPRESSIONS

In this section we show how the analysis of Sections 2.1, 2.2 and 2.3 can be used to enable the numerical value of the driving force B to be determined in various particular cases. The list of driving-force expressions is not exhaustive, but should suffice as an illustration of the method. For concreteness, the "simple reaction" is discussed in terms of combustion; the analysis is of course equally applicable to a liquid-phase reaction, such as that between ammonia and sulphuric acid solutions.

In the interests of brevity, the practical implications of the driving-force expressions are not discussed.

3.1. Transfer of an inert chemical substance

No simplifying assumptions are needed to convert equations (1) and (9) to the standard form of (29) and (30). We can therefore write the driving-force expression for this case, without restriction, as:

$$B = \frac{m_{i,G} - m_{i,S}}{m_{i,S} - m_{i,T}} \quad (38)$$

In this general form the driving force may be used when many substances are transferred

simultaneously, as in rectification of a multi-component mixture, and irrespective of whether some of the other (non- i) components are reacting chemically. Often $m_{i,T}$ must be determined by simultaneous consideration of transfer on the liquid and gas sides of the phase boundary.

Special cases. When i is the *only* mixture component transferred (e.g. $i \equiv \text{H}_2\text{O}$ when steam condenses from a steam-air mixture), we have:

$$m_{i,T} = 1 \quad (39)$$

then

$$B = \frac{m_{i,G} - m_{i,S}}{m_{i,S} - 1} \quad (40)$$

This expression is also valid in many drying, absorption and stripping problems. Clearly, since $m_i \leq 1$, B and m'' are positive when $m_{i,G} \leq m_{i,S}$.

When i is not transferred at all, for example $i \equiv$ air in a drying process, we have:

$$B = \frac{m_{i,G} - m_{i,S}}{m_{i,S}} \quad (41)$$

3.2. Transfer of a chemical element; simplifying assumption (i)

For this case we have, as the driving-force expression:

$$B = \frac{n_{a,G} - n_{a,S}}{n_{a,S} - n_{a,T}} \quad (42)$$

This expression is useful when several chemical reactions take place, as for example in the gas producer. If it is further permissible to assume $\gamma_a = \gamma_b$, any linear combination of n_a and n_b can be used as the conserved property. This fact can be used to derive B -expressions which are easy to evaluate (see [10] for a gas-producer example).

If, further, all γ_a 's are equal and only two streams of substance enter the mixing field (e.g. main stream and transferred substance) it is meaningful to define a quantity f as mass of material from one of the streams per unit mass of mixture, irrespective of the state of chemical aggregation. Since f is a linear combination of the n_a 's, we can write the driving force as:

$$B = \frac{f_G - f_S}{f_S - f_T} \quad (43)$$

This step is one of the foundations of the graphical method of determining B using enthalpy-composition diagrams (see [11, 12, 13, 34] for examples). The first use of such diagrams for transfer calculations was made by Busemann [15] in a pioneer work that has been undeservedly neglected; an important series of contributions to their development has been made by Bosnjakovic [17-20].

3.3. Enthalpy transfer; simplifying assumption (ii)

For this case the relevant driving-force expression is:

$$B = \frac{h_G - h_S}{h_S - h_T} \quad (44)$$

This expression is the second foundation of the graphical method just referred to; it also is particularly useful when complex chemical reactions occur.

Special case. When the wall is impervious to mass transfer it is easy to show, by introducing (44) and letting B tend to zero, that equation (34) degenerates to:

$$\dot{q}''_L = g^* (h_G - h_S) \quad (45)$$

where \dot{q}''_L is the heat flux away from the phase boundary (Fig. 2);

g^* is the value of g valid for $B = 0$.

This expression is useful when calculating heat transfer rates from dissociated gas mixtures, as has recently become widely recognized (see, for example [40]).

3.4. Mass transfer with a simple chemical reaction

Simplifying assumption (iii). Inspection of equation (16) shows that this time we have:

$$B = \frac{(m_{tu} - m_{ox}/r)_G - (m_{tu} - m_{ox}/r)_S}{(m_{tu} - m_{ox}/r)_S - (m_{tu} - m_{ox}/r)_T} \quad (46)$$

Special cases. (a) We consider the case in which one of the reactants, say fuel, is the only transferred substance. In this case it is usual for the main-stream value of the concentration of unburned fuel to be zero. Thus:

$$\left. \begin{aligned} m_{tu,T} &= 1 \\ m_{ox,T} &= 0 \\ m_{tu,G} &= 0 \end{aligned} \right\} \quad (47)$$

Inserting (47) in (46), we obtain:

$$B = \frac{-m_{ox,G}/r - (m_{tu} - m_{ox}/r)_S}{(m_{tu} - m_{ox}/r)_S - 1} \quad (48)$$

(b) If the surface fuel concentration is zero ($m_{tu,S} = 0$), equation (48) reduces to:

$$B = \frac{m_{ox,G}/r - m_{ox,S}/r}{1 + m_{ox,S}/r} \quad (49)$$

This expression is relevant to the combustion of a non-volatile fuel. If, further, the chemical reactivity of the fuel surface is such that $m_{ox,S}$ also equals zero, the driving-force expression of (49) takes up its maximum possible value, namely:

$$B = \frac{m_{ox,G}}{r} \quad (50)$$

Equation (49) is also relevant to the burning of gaseous fuels (transpiration-coolants) which are expelled at such low rates that chemical reaction takes place within the porous walls. (Of course the cooling effect of such a procedure would not be large!)

(c) If the surface oxygen concentration is zero, equation (48) reduces to:

$$B = \frac{m_{ox,G}/r + m_{tu,S}}{1 - m_{tu,S}} \quad (51)$$

This expression usually applies to volatile liquid fuels, and to gaseous fuels injected through the wall at a high rate. However, the expression is seldom useful for determining B ; for the quantity $m_{tu,S}$ is not as a rule among the data of the problem.

(d) A special case of equation (46) arises in the combustion of steel. This may be regarded as a non-volatile fuel which also happens to have a mainly non-volatile oxide. The latter fact leads to an interesting expression for $P_{1,T}$; it is:

$$(m_{tu} - m_{ox}/r)_T = 1 / \{n_{C,steel} (1 + r_{Fe})\} - r_{Fe} \quad (52)$$

where $n_{C,steel}$ is the mass of carbon per unit mass of steel;
 r_{Fe} is the mass of oxidant (usually oxygen) combining with unit mass of iron.

The resulting driving-force expression is:

$$B = \frac{m_{\text{ox},G} - m_{\text{ox},S}}{(n_{\text{Fe}} r_{\text{Fe}} + n_{\text{C,steel}} r_{\text{C}}) / (n_{\text{C,steel}}(1 + r_{\text{Fe}}) - r_{\text{Fe}}) + m_{\text{ox},S}} \quad (53)$$

where r_{C} is the mass of oxidant combining with unit mass of carbon.

The expression on the right-hand side of equation (53) is usually negative and close to the minimum possible value of any B , namely -1 (because $m_{\text{ox},S} \simeq 0$ and $n_{\text{C,steel}}$ is small). It is useful in analyses of the oxygen-cutting process of workshop practice. The process sometimes also occurs inadvertently in rocket motors!

3.5. Mass transfer with a simple chemical reaction

Simplifying assumptions (iv) and (v). Driving-force expressions may also be derived in terms of the conserved properties P_1 and P_2 . They are useful if it is desired to calculate the product concentration at the surface, the mass transfer rate being given.

3.6. Heat transfer without chemical reaction

Simplifying assumption (vi). The driving force expression for this case is obtained most simply if the reference temperature for the fluid-phase enthalpy in equation (22) is taken as t_S . Equation (8) is simplified thereby, and because of the absence of chemical reaction, to:

$$\begin{aligned} h_T &= -\dot{q}''_S / \dot{m}'' \\ &= -Q, \text{ say} \end{aligned} \quad (54)$$

There results:

$$B = \frac{c(t_G - t_S)}{Q} \quad (55)$$

Special cases. (a) Equation (55) is useful in many vaporization, drying and condensation problems. It holds even when the specific heats are not uniform, provided that only small concentration differences occur or that $\gamma = \Gamma$; then c is the specific heat of the mixture of main-stream composition. In *adiabatic vaporization*, $Q = h_{LS}$ where h_{LS} is the latent heat of vaporization of the liquid. So equation (55) reduces to:

$$B = \frac{c(t_G - t_S)}{h_{LS}} \quad (56)$$

(b) In transpiration-cooling by an inert gas, the quantity Q is given by:

$$Q = c_{\text{cool}}(t_S - t_0) \quad (57)$$

where c_{cool} is the constant pressure specific heat of the coolant;

t_0 is the coolant temperature in the reservoir from which it is supplied.

For this case, the driving-force expression becomes

$$B = \frac{c(t_G - t_S)}{c_{\text{cool}}(t_S - t_0)} \quad (58)$$

This shows that, with t_G , t_S and t_0 fixed, B is smaller (and, with it, the necessary coolant flow rate) the larger is the specific heat of the coolant.

3.7. Thermal analysis of mass transfer with a simple chemical reaction

Simplifying assumptions (vii) (viii) and (ix). Restricting consideration to the case in which the only transferred substance is one of the reactants, say fuel, and noting by reference to Fig. 3 that

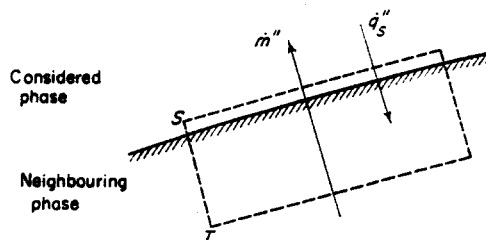


FIG. 3. Control volumes useful in deriving expressions for h_T when only one substance is transferred.

the definition (8) of the enthalpy of the T -state also implies:

$$h_T \dot{m}'' = \sum_j h_{j,L} \dot{m}''_{j,L} - \dot{q}''_L \quad (59)$$

we deduce:

$$h_T = h_{\text{tu},L} - \dot{q}''_L / \dot{m}'' \quad (60)$$

This assumes that only fuel is present on the L -side of the phase boundary.

The driving-force expression resulting from the substitution of (24) and (60) in (33) is:

$$B = \frac{(m_{\text{ox},G} - m_{\text{ox},S}) H / r + c(t_G - t_S)}{h_{LS} + \dot{q}''_L / \dot{m}''} \quad (61)$$

where h_{LS} is the enthalpy increase of the fuel in changing from the L -phase to the S -phase at constant temperature $t_L (= t_S)$.

Equation (61) is appropriate to simplifying assumption (viii).

If simplifying assumption (vii) holds, so that the enthalpy expressed by equation (23) is a conserved property, the driving force expression is found in a similar fashion to be:

$$B = \frac{(m_{tu,G} - m_{tu,S})H + c(t_G - t_S)}{m_{tu,S}H + h_{LS} + \dot{q}''_L/\dot{m}''} \quad (62)$$

A similar expression in terms of m_{prod} can be derived if simplifying assumption (ix) holds.

Special cases. (a) If the surface oxygen content is zero, as is usual for volatile liquid fuels and for burning transpiration-coolants (e.g. hydrogen gas) at high blowing rates, equation (61) reduces to:

$$B = \frac{m_{ox,G}H/r + c(t_G - t_S)}{h_{LS} + \dot{q}''_L/\dot{m}''} \quad (63)$$

This expression is often easy to evaluate.

In the case of transpiration-coolants, when radiant heat transfer is absent, we have:

$$\left. \begin{aligned} h_{LS} &= 0 \\ \dot{q}''_L/\dot{m}'' &= c_{cool}(t_S - t_0) \end{aligned} \right\} \quad (64)$$

then

$$B = \frac{m_{ox,G}H/r + c(t_G - t_S)}{c_{cool}(t_S - t_0)} \quad (65)$$

This should be contrasted with equation (58).

(b) If the surface fuel concentration is zero, as occurs with non-volatile fuels or (sometimes) with combustible transpiration-coolants injected at low rates, and if we can also assume that there is no free fuel in the main stream, equation (62) reduces to:

$$B = \frac{c(t_G - t_S)}{h_{LS} + \dot{q}''_L/\dot{m}''} \quad (66)$$

This expression is sometimes useful for calculating the surface temperature when the driving force B has been previously specified.

(c) If a radiant heat flux is present at the surface, this should clearly be added to \dot{q}''_L with appropriate sign.

3.8. Other forms of driving force

Many other expressions can be derived for the driving force for mass transfer in terms of mixture properties in the G -, S - and T -states. In a particular flow, many of them are valid simultaneously. Where there is a choice, that form of B is to be preferred which is most easily evaluated by reference to known quantities; for example, in determining the mass flow-rate of transpiration-coolant which is necessary to maintain a given surface temperature, the B -expression of (65) is usually preferable to that of (51), since $m_{tu,S}$ is not known.

4. MORE COMPLEX PROBLEMS

The analysis presented in Sections 2 and 3 suffices for the calculation of the mass transfer rate, \dot{m}'' , if sufficient data are available concerning the properties of the G -, S - and T -states. However, the data concerning the latter two states are often only given indirectly. Here we merely list the considerations which must be invoked when these more complex situations are encountered.

(a) Thermodynamic equilibrium at the surface

In many cases, e.g. vaporization, it is permissible to assume that the S -state mixture is in equilibrium with the adjacent phase (L -state). This assumption provides a link between the surface temperature $t_S (= t_L)$ and some of the S -state compositions, e.g. $m_{H_2O,S}$, in a water-vaporization problem. The problem can then be solved by simultaneous use of a B -expression based on mass conservation and the B -expression based on the first law of thermodynamics. Enthalpy-composition diagrams are helpful in the solution [11].

(b) Simultaneous consideration of two phases

Often the data are given in terms of conductances on both sides of a phase boundary; these may involve both mass and heat transfer conductances. Then, using the equality for the two phases of the mass transfer rate and of the T -state (unless surface radiation is present), simultaneous solution of the Ohm's law relation for the two phases permits the determination of \dot{m}'' .

(c) *Reaction-kinetic considerations*

Sometimes chemical reaction rates are insufficiently rapid for thermodynamic equilibrium to prevail. In the case of heterogeneous reactions, a reaction-kinetic relation between \dot{m}'' and the *S*-state must be invoked; this sometimes results in a non-uniform *S*-state, necessitating a return to equations (26) and (27) instead of (29) and (30). In the case of homogeneous reactions at least one equation of the type of equation (2) must be solved.

5. CONCLUSIONS

(a) A standard formulation for the steady convective mass transfer problem has been presented which is capable of accommodating the problems arising in chemical engineering, aeronautical engineering, air-conditioning practice, etc.

(b) The centre-piece of this formulation is the Ohm's law relation, equation (34).

(c) The formulation achieves a separation between aerodynamic factors, incorporated in the conductance *g*, and thermodynamic factors, incorporated in the driving force *B*. The separation is only rendered incomplete by non-uniformity of transport properties in the boundary layer.

(d) The formulation is valid when chemical reaction occurs provided that certain restrictions are placed on the relations between the transport properties.

(e) There is still a great need for standard solutions of the boundary layer equations in the presence of mass transfer.

REFERENCES

1. S. HATTA, *Tech. Rep. Tohoku Imp. Univ.* **8**, 1 (1928-1929); **10**, 119 (1932).
2. T. K. SHERWOOD, *Absorption and Extraction* (1st Ed.). McGraw-Hill, New York (1937).
3. W. NUSSELT, *Z. ver. dtsh. Ing.* **60**, 102 (1916).
4. S. P. BURKE and T. E. W. SCHUMANN, *Proceedings of the Third International Conference on Bituminous Coal* Vol. 2, p. 485 (1931).
5. E. R. G. ECKERT and V. LIEBLEIN, *Forsch. Ing. Wes.* **16**, 33 (1949).
6. D. B. SPALDING, *Fourth Symposium on Combustion* p. 847. Williams and Wilkins, Baltimore (1953).
7. D. B. SPALDING, *Proc. Roy. Soc. A.* **221**, 78 and 100 (1954).
8. H. W. EMMONS, *Z. angew. Math. Mech.* **36**, 60 (1956).
9. D. B. SPALDING, *Proc. Inst. Mech. Engrs.* **168**, 545 (1954).
10. D. B. SPALDING, *Some Fundamentals of Combustion*. Butterworths, London (1955).
11. D. B. SPALDING, Part I. *Chem. Engng. Sci.* **11**, 183-193 (1959); Part II. *Chem. Engng. Sci.* **11**, 225-241 (1959).
12. D. B. SPALDING, *Proc. Inst. Mech. Engrs.* **172**, No. 28 (1958).
13. D. B. SPALDING and R. D. TYLER, in *Conference on Thermodynamic and Transport Properties of Fluids* Session 4, Paper 1. Institute of Mechanical Engineers, London (1957).
14. H. J. MERK, *Appl. Sci. Res. A.* **8**, 73 (1959).
15. A. BUSEMANN, *Der Wärme- und Stoffaustausch*. Julius Springer, Berlin (1933).
16. F. BOSNJAKOVIC, *Technische Thermodynamik*. Steinkopf, Dresden (1937).
17. F. BOSNJAKOVIC, *Wärmediagramme für Vergasung, Verbrennung und Russbildung*. Julius Springer, Berlin (1956).
18. F. BOSNJAKOVIC, *Chem.-Ing.-Techn.* **29**, 187 (1957).
19. F. BOSNJAKOVIC, *Kältetechnik* **9**, 266; **10**, 309 (1957).
20. F. BOSNJAKOVIC, *Kyltekn. Tidskr.* No. 3, 142 (1957).
21. Y. B. ZELDOVICH, and D. A. FRANK-KAMENETSKY, *J. Phys. Chem., Moscow* **12**, 100 (1938).
22. V. A. SHWAB, *Relation between the Temperature and Velocity Fields of the Flame of a Gas Burner*. Gos. Energ. izd., Moscow and Leningrad (1948); cited by Y. B. ZELDOVICH, *Zh. tekhn. fiz.* **19**, 1199 (1949); translated as NACA TM 1296 (1951).
23. G. A. VARSHAVSKII (1945), Burning of liquid fuel drop (diffusion theory), cited by AGAFANOVA *et al.* [25].
24. M. GOLDSMITH and S. S. PENNER, *J. Amer. Rocket Soc.* **24**, 245 (1954).
25. F. A. AGAFANOVA, M. A. GUREVICH and I. I. PALEEV, *Sov. Phys. Tech. Phys.* **2**, 1689 (1958).
26. K. P. COFFIN and R. S. BROKAW, *A General System for Calculating Burning Rates of Particles and Drops and Comparison of Calculated Rates for Carbon, Boron, Magnesium and Iso-octane*. NACA TN 3929 (1957).
27. L. LEES, Convective heat transfer with mass addition and chemical reactions, in *Combustion and Propulsion* p. 451. Third AGARD Colloquium, Pergamon Press, London (1958).
28. G. W. SUTTON, *Seventh Symposium on Combustion* p. 539. Butterworths, London (1959).
29. J. P. HARTNETT and E. R. G. ECKERT, *Mass Transfer Cooling with Combustion in a Laminar Boundary Layer*. Heat Transfer and Fluid Mechanics Institute, Stanford University Press, California (1958).
30. R. BROMBERG and R. P. LIPKIS, *Jet Propulsion* **28**, 659 and 668 (1958).
31. T. K. SHERWOOD and R. L. PIGFORD, *Absorption and Extraction* (2nd Ed.). McGraw-Hill, New York (1952).
32. H. S. MICKLEY, R. C. ROSS, A. L. SQUYERS and W. E. STEWART, *Heat, Mass and Momentum Transfer for Flow over a Flat Plate with Blowing or Suction*. NACA TN 3208 (1954).

33. H. J. MERK, *Appl. Sci. Res.* A 8, 237 (1959).
 34. D. B. SPALDING, *J. Amer. Rocket Soc.* 29, 666 (1959).
 35. J. N. B. LIVINGOOD and P. L. DONOUGHE, *Summary of Laminar Boundary Layer Solutions for Wedge-type Flow over Convection and Transpiration-cooled Surfaces.* NACA TN 3588 (1955).
 36. G. S. AMBROK, *Sov. Phys. Tech. Phys.* 2, 1979 (1957).
 37. H. S. MICKLEY and R. S. DAVIS, *Momentum Transfer for Flow over a Flat Plate with Blowing.* NACA TN 4017 (1957).
 38. M. W. RUBESIN, *An Analytical Estimation of the Effect of Transpiration Cooling on the Heat-transfer and Skin-friction Characteristics of a Compressible Turbulent Boundary Layer.* NACA TN 3341 (1954).
 39. M. W. RUBESIN and C. C. PAPPAS, *An Analysis of the Turbulent Boundary-layer Characteristics on a Flat Plate with Distributed Light-gas Injection.* NACA TN 4149 (1958).
 40. J. A. FAY and F. R. RIDDELL, *J. Aero. Space Sci.* 25, 73 (1958).
 41. D. B. SPALDING and A. G. SMITH, *Brennst.-Wärmechr.* 10, 271 (1958).

- V = velocity of fluid mixture defined as G/ρ (ft/hr);
 μ = viscosity of fluid mixture (lb/ft hr);
 g_0 = constant in Newton's Second Law of motion,
 = 4.1×10^8 (lb ft/Lb hr²);
 J = mechanical equivalent of heat
 = 778 (ft Lb/B.t.u.);
 R_S = vectorial radius of curvature of local stream line;
 R_S = scalar magnitude of R_S (ft).

Equation (A.1) simplifies in the same manner as equation (4) provided that $\gamma_j = \Gamma$ for all j and further that (a) we also have $\Gamma = \mu$ (Prandtl number = 1), and that (b) the radius of curvature of the steamlines R_S is large. The resulting simplified equation is:

$$G. (\nabla h^0) - \nabla \{ \Gamma (\nabla h^0) \} = 0 \quad (A.2)$$

This equation has the standard form. It differs from equation (14) only in that h^0 replaces h .

The wall-flux boundary conditions for equations (A.1) and (A.2) are identical with those for equations (4) and (14); for the fluid kinetic energy is negligible at the wall, so that h and h^0 are identical. It follows that the stagnation enthalpy h^0 is a "conserved property of the second class"; it may therefore be used for the derivation of a driving-force expression.

The simplification of the energy equation which results when $Pr = 1$ is of course well known, being usually attributed to Crocco. The present proof appears to be rather simpler and more general than those usually given.

APPENDIX

The Energy Equation Including Kinetic Energy and Shear Work

A more complete form of equation (4) is

$$G. (\nabla h^0) - \nabla \{ \sum_j \gamma_j h_j (\nabla m_j) \} - \nabla \cdot \{ \Gamma c (\nabla t) \} - \nabla \cdot \left\{ \frac{\mu}{g_0 J} \left[\nabla (V^2/2) - \frac{V^2}{R_S} \cdot \frac{R_S}{R_S} \right] \right\} = 0 \quad (A.1)$$

where

$$h^0 = \text{stagnation enthalpy of fluid mixture} \\ = h + V^2/2g_0J \text{ (B.t.u./lb);}$$